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FAKULTA INFORMATIKY A STATISTIKY
Katedra statistiky a pravděpodobnosti

STATISTIKA

VZORCE PRO 4ST201

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Popisná statistika

$$p_i = \frac{n_i}{n} \quad \sum_{i=1}^k n_i = n \quad \sum_{i=1}^k p_i = 1 \quad i = 1, 2, \dots, k$$

$$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$$

$$\tilde{x}_P, 0 < P < 1 \quad x_{(z_P)}, \quad nP < z_P < nP + 1,$$

$$\frac{x_{(z_P)} + x_{(z_P+1)}}{2}, nP = z_P$$

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

$$\bar{x} = \frac{\sum_{i=1}^k n_i x_i}{\sum_{i=1}^k n_i}$$

$$\bar{x} = \sum_{i=1}^k p_i x_i$$

$$\bar{x}_H = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$$

$$\bar{x}_H = \frac{\sum_{i=1}^k n_i}{\sum_{i=1}^k \frac{n_i}{x_i}}$$

$$\bar{x}_H = \frac{1}{\sum_{i=1}^k \frac{p_i}{x_i}}$$

$$\bar{x}_G = \sqrt[n]{\prod_{i=1}^n x_i} = \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}$$

$$R = x_{\max} - x_{\min}$$

$$s_x^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$$

$$s_x^2 = \overline{x^2} - \bar{x}^2 = \frac{\sum_{i=1}^n x_i^2}{n} - \left(\frac{\sum_{i=1}^n x_i}{n} \right)^2$$

$$s_x^2 = \frac{\sum_{i=1}^k n_i (x_i - \bar{x})^2}{\sum_{i=1}^k n_i}$$

$$s_x^2 = \overline{x^2} - \bar{x}^2 = \frac{\sum_{i=1}^k n_i x_i^2}{\sum_{i=1}^k n_i} - \left(\frac{\sum_{i=1}^k n_i x_i}{\sum_{i=1}^k n_i} \right)^2$$

$$s_x^2 = \sum_{i=1}^k p_i (x_i - \bar{x})^2$$

$$s_x^2 = \overline{x^2} - \bar{x}^2 = \sum_{i=1}^k p_i x_i^2 - \left[\sum_{i=1}^k p_i x_i \right]^2$$

$$s_x^2 = \overline{s^2} + s_{\bar{x}}^2 = \frac{\sum_{i=1}^k n_i s_i^2}{\sum_{i=1}^k n_i} + \frac{\sum_{i=1}^k n_i (\bar{x}_i - \bar{x})^2}{\sum_{i=1}^k n_i}$$

$$\bar{x} = \frac{\sum_{i=1}^k n_i \bar{x}_i}{\sum_{i=1}^k n_i}$$

$$s_x^2 = \sum_{i=1}^k p_i s_i^2 + \sum_{i=1}^k p_i (\bar{x}_i - \bar{x})^2$$

$$\bar{x} = \sum_{i=1}^k p_i \bar{x}_i$$

$$s_x = \sqrt{s_x^2}$$

$$V_x = \frac{s_x}{\bar{x}}$$

Pravděpodobnost

Počet pravděpodobnosti

$$P(A) = \frac{m}{n} \quad P(\bar{A}) = 1 - P(A)$$

$$P(A \cap B) \leq \min(P(A), P(B)) \quad P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad P(A \cup B) = P(A) + P(B)$$

Náhodné veličiny

$$P(x) = P(X = x) \quad F(x) = P(X \leq x) = \sum_{x_i \leq x} P(x_i) \quad P(x_1 < X \leq x_2) = \sum_{x_1 < x_i \leq x_2} P(x_i) = F(x_2) - F(x_1)$$

$$E(X) = \sum_x x P(x) \quad D(X) = \sum_x x^2 P(x) - \left[\sum_x x P(x) \right]^2 \quad \sigma = \sqrt{D(X)}$$

$$f(x) = F'(x) \quad \int_{-\infty}^{\infty} f(x) dx = 1 \quad F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

$$P(x_1 < X \leq x_2) = F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x) dx$$

$$x_p, 0 < P < 1 \quad F(x_p) = P \quad x_p = F^{-1}(P)$$

$$E(X) = \int_{-\infty}^{\infty} xf(x) dx \quad D(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - \left[\int_{-\infty}^{\infty} xf(x) dx \right]^2 \quad \sigma = \sqrt{D(X)}$$

Pravděpodobnostní rozdělení

Alternativní rozdělení $A[\pi]$

$$P(x) = \pi^x (1 - \pi)^{1-x} \quad x = 0, 1, \quad 0 < \pi < 1$$

$$E(X) = \pi \quad D(X) = \pi(1 - \pi)$$

Binomické rozdělení $Bi[n; \pi]$

$$P(x) = \binom{n}{x} \pi^x (1 - \pi)^{n-x} \quad x = 0, 1, 2, \dots, n, \quad 0 < \pi < 1,$$

$$E(X) = n\pi \quad D(X) = n\pi(1 - \pi)$$

Poissonovo rozdělení $Po[\lambda]$

$$P(x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad x = 0, 1, \dots, \lambda > 0,$$

$$E(X) = \lambda \quad D(X) = \lambda$$

Hypergeometrické rozdělení $Hy[N;M;n]$

$$P(x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}}, x = \max(0, M - N + n), \dots, \min(M, n), n > 0, N \geq n, M \leq N$$

$$E(X) = n \frac{M}{N} \quad D(X) = n \frac{M}{N} \left(1 - \frac{M}{N}\right) \frac{N-n}{N-1}$$

Normované normální rozdělení $N[0;1]$

$$-\infty < u < \infty, \quad \Phi(u) = 1 - \Phi(-u) \quad u_p = -u_{1-p}$$

$$E(U) = 0 \quad D(U) = 1$$

Normální rozdělení $N[\mu; \sigma^2]$

$$-\infty < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0 \quad E(X) = \mu \quad D(X) = \sigma^2$$

$$u = \frac{x - \mu}{\sigma} \quad F(x) = \Phi(u) = \Phi\left(\frac{x - \mu}{\sigma}\right) \quad x_p = \mu + \sigma u_p$$

$$P(x_1 < X \leq x_2) = P\left(\frac{x_1 - \mu}{\sigma} < \frac{X - \mu}{\sigma} \leq \frac{x_2 - \mu}{\sigma}\right) = P(u_1 < U \leq u_2) = \Phi(u_2) - \Phi(u_1)$$

Chi-kvadrát rozdělení $\chi^2[\nu]$ $x > 0, \nu \in N$

Rozdělení t (Studentovo) $t[\nu]$ $-\infty < t < \infty, \nu \in N \quad t_p[\nu] = -t_{1-p}[\nu]$

F - rozdělení (Fisherovo- Snedecorovo) $F[\nu_1; \nu_2]$ $x > 0, \nu_1, \nu_2 \in N,$

$$F_p[\nu_1; \nu_2] = \frac{1}{F_{1-p}[\nu_2; \nu_1]}$$

Matematická statistika

$$s'_x = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}$$

Odhady parametrů

střední hodnota est $\mu = \hat{\mu} = \bar{x}$ est $N\mu = N\bar{x}$

normální rozdělení

a) σ^2 známé

$$P\left(\bar{x} - u_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + u_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(\bar{x} - u_{1-\alpha} \frac{\sigma}{\sqrt{n}} < \mu\right) = 1 - \alpha, \quad P\left(\mu < \bar{x} + u_{1-\alpha} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

b) σ^2 neznámé

$$P\left(\bar{x} - t_{1-\alpha/2}[n-1] \frac{s'_x}{\sqrt{n}} < \mu < \bar{x} + t_{1-\alpha/2}[n-1] \frac{s'_x}{\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(\bar{x} - t_{1-\alpha}[n-1] \frac{s'_x}{\sqrt{n}} < \mu\right) = 1 - \alpha, \quad P\left(\mu < \bar{x} + t_{1-\alpha}[n-1] \frac{s'_x}{\sqrt{n}}\right) = 1 - \alpha$$

obecné rozdělení, σ^2 neznámé, velký výběr ($n > 30$)

$$P\left(\bar{x} - u_{1-\alpha/2} \frac{s'_x}{\sqrt{n}} < E(X) < \bar{x} + u_{1-\alpha/2} \frac{s'_x}{\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(\bar{x} - u_{1-\alpha} \frac{s'_x}{\sqrt{n}} < E(X)\right) = 1 - \alpha, \quad P\left(E(X) < \bar{x} + u_{1-\alpha} \frac{s'_x}{\sqrt{n}}\right) = 1 - \alpha$$

rozptyl σ^2 (normální rozdělení) est $\sigma^2 = \hat{\sigma}^2 = s_x'^2$

Parametr π alternativního rozdělení (odhad relativní četnosti základního souboru)

est $\pi = \hat{\pi} = p$ est $N\pi = Np$

$$P\left(p - u_{1-\alpha/2} \sqrt{\frac{p(1-p)}{n}} < \pi < p + u_{1-\alpha/2} \sqrt{\frac{p(1-p)}{n}}\right) = 1 - \alpha$$

$$P\left(p - u_{1-\alpha} \sqrt{\frac{p(1-p)}{n}} < \pi\right) = 1 - \alpha, \quad P\left(\pi < p + u_{1-\alpha} \sqrt{\frac{p(1-p)}{n}}\right) = 1 - \alpha$$

Testování hypotéz

Střední hodnota normálního rozdělení

H ₀	H ₁	Testové kritérium	Kritický obor
$\mu = \mu_0$	$\mu > \mu_0$	σ^2 známé $U = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} \quad U \sim N[0;1]$	$W_\alpha = \{u; u \geq u_{1-\alpha}\}$
	$\mu < \mu_0$		$W_\alpha = \{u; u \leq -u_{1-\alpha}\}$
	$\mu \neq \mu_0$	σ^2 neznámé $T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s'_x} \quad T \sim t[n-1]$	$W_\alpha = \{u; u \geq u_{1-\alpha/2}\}$
			$W_\alpha = \{t; t \geq t_{1-\alpha}\}$
			$W_\alpha = \{t; t \leq -t_{1-\alpha}\}$
			$W_\alpha = \{t; t \geq t_{1-\alpha/2}\}$

Střední hodnota, obecné rozdělení, velký výběr

H ₀	H ₁	Testové kritérium	Kritický obor
$E(X) = \mu_0$	$E(X) > \mu_0$	σ^2 neznámé ($n > 30$) $U = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s'_x} \quad U \approx N[0;1]$	$W_\alpha = \{u; u \geq u_{1-\alpha}\}$
	$E(X) < \mu_0$		$W_\alpha = \{u; u \leq -u_{1-\alpha}\}$
	$E(X) \neq \mu_0$		$W_\alpha = \{u; u \geq u_{1-\alpha/2}\}$

Parametr π alternativního rozdělení (velké výběry)

H ₀	H ₁	Testové kritérium	Kritický obor
$\pi = \pi_0$	$\pi > \pi_0$	$U = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}} \quad U \approx N[0;1]$	$W_\alpha = \{u; u \geq u_{1-\alpha}\}$
	$\pi < \pi_0$		$W_\alpha = \{u; u \leq -u_{1-\alpha}\}$
	$\pi \neq \pi_0$		$W_\alpha = \{u; u \geq u_{1-\alpha/2}\}$

Rovnost středních hodnot dvou rozdělení

velké nezávislé výběry

H ₀	H ₁	Testové kritérium	Kritický obor
$\mu_1 = \mu_2$	$\mu_1 > \mu_2$	σ_1^2, σ_2^2 neznámé $U = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1'^2}{n_1} + \frac{s_2'^2}{n_2}}} \quad U \approx N[0;1]$	$W_\alpha = \{u; u \geq u_{1-\alpha}\}$
$(\mu_1 - \mu_2 = 0)$	$\mu_1 < \mu_2$		$W_\alpha = \{u; u \leq -u_{1-\alpha}\}$
	$\mu_1 \neq \mu_2$		$W_\alpha = \{u; u \geq u_{1-\alpha/2}\}$
$E(X_1) = \mu_1, E(X_2) = \mu_2$			
$D(X_1) = \sigma_1^2, D(X_2) = \sigma_2^2$			

závislé výběry z normálního rozdělení (párový t-test)

H ₀	H ₁	Testové kritérium	Kritický obor
$\mu_1 = \mu_2$	$\mu_1 > \mu_2$	$T = \frac{\sqrt{n} \bar{d}}{s'_d} \quad T \sim t[n-1]$ $d_i = x_{1i} - x_{2i}, i = 1, 2, \dots, n$	$W_\alpha = \{t; t \geq t_{1-\alpha}\}$
$(\mu_1 - \mu_2 = 0)$	$\mu_1 < \mu_2$		$W_\alpha = \{t; t \leq -t_{1-\alpha}\}$
	$\mu_1 \neq \mu_2$		$W_\alpha = \{t; t \geq t_{1-\alpha/2}\}$

Chí-kvadrát test dobré shody

H ₀ a H ₁	Testové kritérium	Kritický obor
H ₀ : $\pi_i = \pi_{0,i} \quad i = 1, \dots, k$ H ₁ : non H ₀	$G = \sum_{i=1}^k \frac{(n_i - n\pi_{0,i})^2}{n\pi_{0,i}} \quad G \approx \chi^2[k-1]$	$W_\alpha = \{G; G \geq \chi^2_{1-\alpha}\}$ $n\pi_{0,i} \geq 5$

Analýza závislostí

Kontingenční tabulka ($r \times s$)

$$n_{i.} = \sum_{j=1}^s n_{ij} \quad n_{.j} = \sum_{i=1}^r n_{ij} \quad n'_{ij} = \frac{n_{i.} n_{.j}}{n} \quad n'_{ij} \geq 5$$

H ₀	H ₁	Testové kritérium	Kritický obor
$\pi_{ij} = \pi_{i.} \pi_{.j}$ $1 \leq i \leq r$ $1 \leq j \leq s$	non H ₀	$G = \sum_{i=1}^r \sum_{j=1}^s \frac{(n_{ij} - n'_{ij})^2}{n'_{ij}} \quad G \approx \chi^2 [(r-1)(s-1)]$	$W_\alpha = \{G; G \geq \chi^2_{1-\alpha}\}$

$$C = \sqrt{\frac{G}{n+G}} \quad V = \sqrt{\frac{G}{n(m-1)}}, \quad m = \min(r, s)$$

$$r = s = 2 \quad G = n \frac{(n_{11}n_{22} - n_{12}n_{21})^2}{n_{1.}n_{2.}n_{.1}n_{.2}}$$

Analýza rozptylu

$$S_y = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2 = S_{y.m} + S_{y.v} \quad S_{y.m} = \sum_{i=1}^k n_i (\bar{y}_i - \bar{y})^2 \quad S_{y.v} = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$$

$$P^2 = \frac{S_{y.m}}{S_y}$$

H ₀	H ₁	Testové kritérium	Kritický obor
$\mu_1 = \mu_2 = \dots = \mu_k$	non H ₀	$F = \frac{\frac{S_{y.m}}{k-1}}{\frac{S_{y.v}}{n-k}} \quad F \sim F[k-1; n-k]$	$W_\alpha = \{F; F \geq F_{1-\alpha}\}$

Regrese a korelace

regresní přímka $y = \beta_0 + \beta_1 x + \varepsilon, \quad Y = b_0 + b_1 x \quad \min_{b_0, b_1} \sum_{i=1}^n (y_i - b_0 - b_1 x_i)^2$

$$s_{xy} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{n} = \bar{xy} - \bar{x} \cdot \bar{y}$$

$$b_1 = b_{yx} = \frac{n \sum y_i x_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} = \frac{\bar{xy} - \bar{x} \cdot \bar{y}}{x^2 - \bar{x}^2} = \frac{s_{xy}}{s_x^2}$$

$$b_0 = \frac{\sum y_i \sum x_i^2 - \sum y_i x_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2} = \bar{y} - b_{yx} \bar{x}$$

Jiné regresní funkce $Y = b_0 + b_1x + b_2x^2$ $Y = b_0 + b_1x_1 + b_2x_2 + \dots + b_kx_k$

$$S_y = \sum_{i=1}^n (y_i - \bar{y})^2 \quad S_T = \sum_{i=1}^n (Y_i - \bar{y})^2 \quad S_R = \sum_{i=1}^n (y_i - Y_i)^2 = \sum_{i=1}^n e_i^2$$

$$S_y = S_R + S_T \quad s_R^2 = \frac{S_R}{n-p} \quad s_R = \sqrt{\frac{S_R}{n-p}} = \sqrt{s_R^2}$$

$$R^2 = I^2 = \frac{S_T}{S_y} \quad I_{ADJ}^2 = R_{ADJ}^2 = 1 - (1 - I^2) \frac{n-1}{n-p}$$

Test hypotézy o regresních parametrech

H ₀	H ₁	Testové kritérium	Kritický obor
$\beta_i = 0$	$\beta_i \neq 0$	$T = \frac{b_i}{s(b_i)} \quad T \sim t[n-p]$	$W_\alpha = \{t; t \geq t_{1-\alpha/2}\}$

Test o modelu ($p = k + 1$)

H ₀	H ₁	Testové kritérium	Kritický obor
$\beta_0 = c$ $\beta_1 = 0$... $\beta_k = 0$	non H ₀	$F = \frac{\frac{S_T}{p-1}}{\frac{S_R}{n-p}} \quad F \sim F[p-1; n-p]$	$W_\alpha = \{F; F \geq F_{1-\alpha}\}$

Korelační koeficient

$$r_{yx} = r_{xy} = \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{\sqrt{n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \sqrt{n \sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n y_i\right)^2}} = \frac{\bar{xy} - \bar{x} \bar{y}}{\sqrt{(\bar{x^2} - \bar{x}^2)(\bar{y^2} - \bar{y}^2)}} = \frac{s_{xy}}{s_x s_y}$$

Test o nulovosti korelačního koeficientu

H ₀	H ₁	Testové kritérium	Kritický obor
$\rho_{yx} = 0$	$\rho_{yx} \neq 0$	$T = \frac{r_{yx} \sqrt{n-2}}{\sqrt{1-r_{yx}^2}} \quad T \sim t[n-2]$	$W_\alpha = \{t; t \geq t_{1-\alpha/2}\}$

Časové řady

$$\bar{y} = \frac{\sum_{t=1}^n y_t}{n} \quad \bar{y} = \frac{\frac{1}{2} y_1 + \sum_{t=2}^{n-1} y_t + \frac{1}{2} y_n}{n-1} \quad \bar{y} = \frac{\frac{y_1 + y_2}{2} d_1 + \frac{y_2 + y_3}{2} d_2 + \dots + \frac{y_{n-1} + y_n}{2} d_{n-1}}{d_1 + d_2 + \dots + d_{n-1}}$$

$$\Delta_t = y_t - y_{t-1} \quad \bar{\Delta} = \frac{1}{n-1} \sum_{t=2}^n \Delta_t = \frac{y_n - y_1}{n-1}$$

$$k_t = \frac{y_t}{y_{t-1}} \quad \bar{k} = \sqrt[n-1]{k_2 k_3 \dots k_n} = \sqrt[n-1]{\frac{y_n}{y_1}}$$

$$I_{t/1} = \frac{y_t}{y_1} = I_{2/1} I_{3/2} \dots I_{t/t-1} \quad I_{t/t-1} = \frac{y_t}{y_{t-1}} = \frac{I_{t/1}}{I_{t-1/1}}$$

Klouzavé průměry

$$m = 2p + 1 \quad \bar{y}_t = \frac{\sum_{i=-p}^p y_{t+i}}{m} = \frac{y_{t-p} + \dots + y_{t-1} + y_t + y_{t+1} + \dots + y_{t+p}}{m}$$

$$m = 2p \quad \bar{y}_t = \frac{1}{2m} (y_{t-p} + 2y_{t-p+1} + \dots + 2y_{t-1} + 2y_t + 2y_{t+1} + \dots + 2y_{t+p-1} + y_{t+p})$$

Dekompozice časové řady

$$y_t = T_t + S_t + C_t + \varepsilon_t \quad y_t = T_t S_t C_t \varepsilon_t$$

$$T_t = \beta_0 + \beta_1 t \quad \hat{T}_t = b_0 + b_1 t \quad T_t = \beta_0 + \beta_1 t + \beta_2 t^2 \quad \hat{T}_t = b_0 + b_1 t + b_2 t^2$$

$$MSE = \frac{\sum_{t=1}^n (y_t - \hat{T}_t)^2}{n} = \frac{\sum_{t=1}^n e_t^2}{n} \quad D = \frac{\sum_{t=2}^n (e_t - e_{t-1})^2}{\sum_{t=1}^n e_t^2}$$

Regresní metoda s umělými proměnnými (lineární trend, sezónnost délky 4)

$$y_t = T_t + S_t + \varepsilon_t = \beta_0 + \beta_1 t + \alpha_1 x_{1t} + \alpha_2 x_{2t} + \alpha_3 x_{3t} + \varepsilon_t$$

$$\bar{a} = \frac{a_1 + a_2 + a_3}{4} \quad S_{i+4j} = a_i - \bar{a}, i = 1, 2, 3 \quad S_{4+4j} = -\bar{a} \quad \hat{T}_t = (b_0 + \bar{a}) + b_1 t$$

Indexní analýza

$$Q = pq \quad IQ = Ip \cdot Iq$$

$$Ip = \frac{p_1}{p_0} \quad \Delta p = p_1 - p_0 \quad Iq = \frac{q_1}{q_0} \quad \Delta q = q_1 - q_0 \quad IQ = \frac{Q_1}{Q_0} \quad \Delta Q = Q_1 - Q_0$$

$$I(\Sigma q) = \frac{\sum q_1}{\sum q_0} = \frac{\sum Iq \cdot q_0}{\sum q_0} = \frac{\sum q_1}{\sum \frac{q_1}{Iq}} \quad \Delta(\Sigma q) = \sum q_1 - \sum q_0$$

$$I(\Sigma Q) = \frac{\sum Q_1}{\sum Q_0} = \frac{\sum p_1 q_1}{\sum p_0 q_0} = \frac{\sum IQ \cdot Q_0}{\sum Q_0} = \frac{\sum Q_1}{\sum \frac{Q_1}{IQ}} \quad \Delta(\Sigma Q) = \sum Q_1 - \sum Q_0$$

$$\bar{p} = \frac{\bar{p}_1}{\bar{p}_0} = \frac{\frac{\sum Q_1}{\sum q_1}}{\frac{\sum Q_0}{\sum q_0}} = \frac{\sum p_1 q_1}{\sum p_0 q_0} = \frac{\sum \frac{Q_1}{p_1}}{\sum \frac{Q_0}{p_0}} \quad \Delta \bar{p} = \bar{p}_1 - \bar{p}_0 = \frac{\sum p_1 q_1}{\sum q_1} - \frac{\sum p_0 q_0}{\sum q_0}$$

$$Ip^{(L)} = \frac{\sum p_1 q_0}{\sum p_0 q_0} = \frac{\sum Ip \cdot p_0 q_0}{\sum p_0 q_0} = \frac{\sum Ip \cdot Q_0}{\sum Q_0} \quad Ip^{(P)} = \frac{\sum p_1 q_1}{\sum p_0 q_1} = \frac{\sum p_1 q_1}{\sum \frac{p_1 q_1}{Ip}} = \frac{\sum Q_1}{\sum \frac{Q_1}{Ip}}$$

$$Ip^{(F)} = \sqrt{Ip^{(L)} \cdot Ip^{(P)}}$$

$$Iq^{(L)} = \frac{\sum p_0 q_1}{\sum p_0 q_0} = \frac{\sum Iq \cdot p_0 q_0}{\sum p_0 q_0} = \frac{\sum Iq \cdot Q_0}{\sum Q_0} \quad Iq^{(P)} = \frac{\sum p_1 q_1}{\sum p_1 q_0} = \frac{\sum p_1 q_1}{\sum \frac{p_1 q_1}{Iq}} = \frac{\sum Q_1}{\sum \frac{Q_1}{Iq}}$$

$$Iq^{(F)} = \sqrt{Iq^{(L)} \cdot Iq^{(P)}}$$